5.2 Orthogonal Complements and Orthogonal Projections.

Def. Consider a subspace W of $\mathbb{R}^n$

then,

a vector $\vec{v}$ in $\mathbb{R}^n$ is orthogonal to

the Subspace W, if $\vec{V} \cdot \vec{\omega} = 0$, for all $\vec{\omega} \in W$

Now, we define

$$W \text{ orthogonal} = W^{\perp} = \left\{ \vec{V} \in \mathbb{R}^n : \vec{V} \cdot \vec{\omega} = 0 \text{ for all } \vec{\omega} \in W \right\}$$

Exercises:

1) Find $W^\perp$ define it for W given by

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y - 2x = 0 \right\}$$ First find a basis.

Answer:

A basis for W is given by

$y = 2x \qquad \vec{x} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$B_W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Now, $\vec{x} \in W^\perp$ if (why?)

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \iff x + 2y = 0$$

Then $W = \left\{ \text{All } \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}$.

2) If W is the plane: $3x + 2y + z = 0$ find $W^\perp$. First find a basis.

$z = -3x - 2y$

Basis for W

$$\Rightarrow \vec{x} = \begin{bmatrix} x \\ y \\ -3x-2y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \qquad B_W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

$\vec{x}$ is in $W^\perp$ if

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 0 \iff x + 3z = 0 \Rightarrow x = -3z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 0 \iff y - 2z = 0 \Rightarrow y = 2z$$

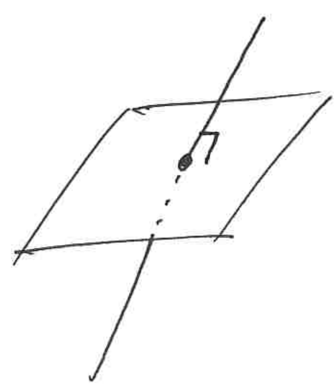
$$\vec{x} = \begin{bmatrix} -3z \\ 2z \\ z \end{bmatrix} = z \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

So a basis for $W^\perp$ is

$$B_{W^\perp} = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and $W^\perp$ is the line

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \; t \in \mathbb{R} \right\}$$



Thm 5.9 $\vec{W} \subset \mathbb{R}^n$ is a Subspace.

a) $W^\perp$ is a Subspace.

b) $(W^\perp)^\perp = W$.

c) $W \cap W^\perp = \{\vec{0}\}$.

d) If $W = \text{Span}(\vec{w}_1, .., \vec{w}_k)$

Then, $\vec{v}$ is in $W^\perp$ if and only if

$\vec{v} \cdot \vec{w}_i = 0$, for all $i = 1, .., k$.

Verify this properties with our previous examples.

thm 5.11

Let W be a Subspace of $\mathbb{R}^n$
and $\vec{v}$ in $\mathbb{R}^n$, then

$$\vec{v} = \vec{w} + \vec{w}^{\perp}$$

where $\vec{w}$ is in W and $\vec{w}^{\perp}$ is in $W^{\perp}$.
and this decomposition of $\vec{v}$ is unique.

Corollary 5.12 W is a Subspace of $\mathbb{R}^n$,
then $(W^{\perp})^{\perp} = W$.

Proof. If $\vec{w}$ is in W and $\vec{u}$ is in $W^{\perp}$
then $\vec{w} \cdot \vec{u} = 0 \Rightarrow \vec{w}$ is in $(W^{\perp})^{\perp}$.
$W \subset (W^{\perp})^{\perp}$.

Also, if $\vec{z}$ is in $(W^{\perp})^{\perp} \Rightarrow$
$\vec{z} = \vec{w} + \vec{w}^{\perp}$

then

$$0 = \vec{z} \cdot \vec{w}^{\perp} = \cancel{\vec{w} \cdot \vec{w}^{\perp}} + \vec{w}^{\perp} \cdot \vec{w}^{\perp} = \vec{w}^{\perp} \cdot \vec{w}^{\perp} \Rightarrow \vec{w}^{\perp} = \vec{0}$$

and $z = \vec{w}$ is in W.
$(W^{\perp})^{\perp} \subset W$.

<u>Thm 5.13</u> W subspace of $\mathbb{R}^n$

then

$$\dim W + \dim W^{\perp} = n.$$

<u>Proof.</u> If $\{u_1, \dots, u_k\}$ is an orthogonal basis for W

and $\{v_1, \dots, v_\ell\}$ is an orthog. basis for $W^\perp$.

then $\dim W = k$, $\dim W^\perp = \ell$.

and $B = \{u_1, \dots, u_k, v_1 \dots, v_\ell\}$ is a basis for $\mathbb{R}^n$.

$u_i \cdot v_j = 0$ for $1 \le i \le k$, $1 \le j \le \ell$.

So B is lin. indep.

Also, any v in $\mathbb{R}^n$

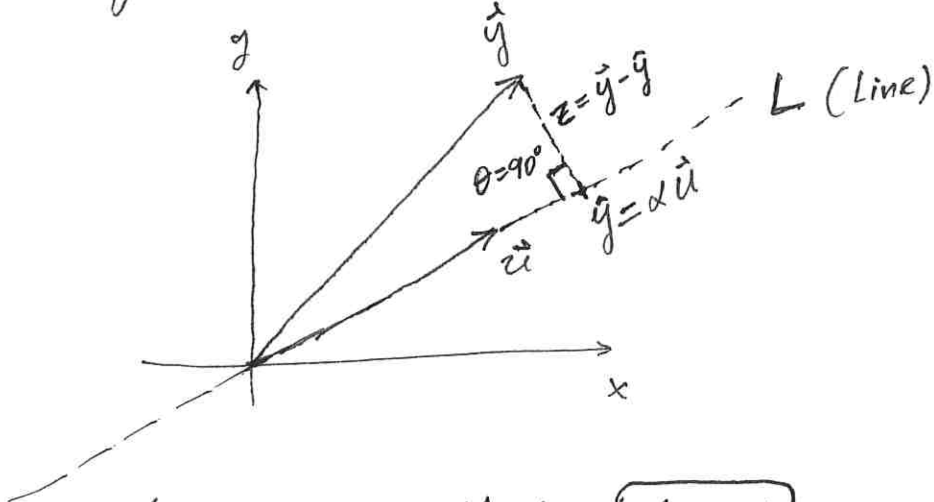
$\vec{v} = \vec{w} + \vec{w}^{\perp} = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_\ell \vec{v}_\ell$

$\Rightarrow$ ~~B~~ $\text{span}(B) = \mathbb{R}^n$.

and B is a basis.

Orthogonal Projections

Consider $\vec{y}, \vec{u} \in \mathbb{R}^2$ as in the graph below



We want to obtain α such that $\boxed{\hat{y} = \alpha \vec{u}}$ and $\boxed{\vec{z} = \vec{y} - \hat{y}}$ is orthogonal to $\vec{u}$. (6.1)

From our previous results and definitions

$\vec{z} = \vec{y} - \hat{y}$ is orthogonal to $\vec{u}$ if and only if

$$(\vec{y} - \hat{y}) \cdot \vec{u} = 0$$

This implies

$$0 = (\vec{y} - \alpha \vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - (\alpha \vec{u}) \cdot \vec{u} \overset{\text{Thm 1}}{=} \vec{y} \cdot \vec{u} - \alpha (\vec{u} \cdot \vec{u})$$

$$\Rightarrow \quad \alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

or

$$\boxed{\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}} \qquad (6.2)$$

Combining (6.1) and (6.2), we have obtained a decomposition of $\vec{y} \in \mathbb{R}^2$ as

$$\boxed{\vec{y} = \hat{y} + \vec{z}} \tag{7.1}$$

where $\hat{y} = \left(\dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}\right) \in \text{Span}\{\vec{u}\}$ and $z \in (\text{Span}\{\vec{u}\})^{\perp}$

<u>Definition</u> The vector $\hat{y}$ given by

$$\boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$$

is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$

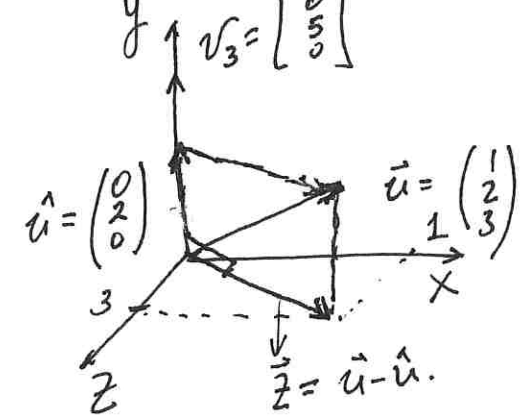
It is also denoted as

$$\boxed{\hat{y} = \text{proj}_{\vec{u}}\,\vec{y}}$$

As it will be shown later this decomposition (7.1) is unique.

Exercise For the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$



Find the unique decomposition

$$\vec{u} = \hat{u} + \vec{z}$$

where $\hat{u} = \text{proj}_{\vec{v}_3} \vec{u}$ and z is orthogonal to $\vec{v}_3$.

Answ:

According to (7.1)

$$\hat{u} = \text{proj}_{\vec{v}_3} \vec{u} = \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3 = \frac{[0\ 5\ 0] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{[0\ 5\ 0] \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$= \frac{2}{5} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

and

$$\vec{z} = \vec{u} - \hat{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Therefore, the decomposition of $\vec{u}$ is as follows:

$$\vec{u} = \overset{\hat{u}}{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}} + \overset{\vec{z}}{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Verification that $\vec{z}$ is orthogonal to $\vec{v}_3$

$$\vec{z} \cdot \vec{v}_3 = [1\ 0\ 3] \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = 0 \checkmark$$

~~And~~ $\vec{z} = \vec{u} - \hat{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$

Verification of $\vec{z}$ orthogonal to $\vec{v}_3$

$$\vec{z} \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = 0 \checkmark$$

or

$$\boxed{\begin{array}{c} \vec{u} = \hat{u} + \vec{z} \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{array}}$$

The concept of orthogonal projection can be extended to any subspace $W \subset \mathbb{R}^n$ (It does not have to be a line along $\vec{u}$).

<u>Thm 8</u> (Orthogonal Decomposition Theorem)

$W \subset \mathbb{R}^n$ subspace and $B = \{\vec{u}_1, \dots, \vec{u}_p\}$ orthogonal basis of W.

Then,

Any $\vec{y} \in \mathbb{R}^n$ can be written uniquely as

$$\boxed{\vec{y} = \hat{y} + \vec{z}} \qquad (8.1)$$

where $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$

and

$$\boxed{\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p = \mathrm{proj}_W \vec{y}} \qquad (8.2)$$

<u>Proof.</u>

Given $B = \{\vec{u}_1, .., \vec{u}_p\}$ orthog. basis of Subspace W.

We want to find $\hat{y}$ in W and $\vec{z} \in W^{\perp}$ such that

$$\vec{y} = \hat{y} + \vec{z}$$

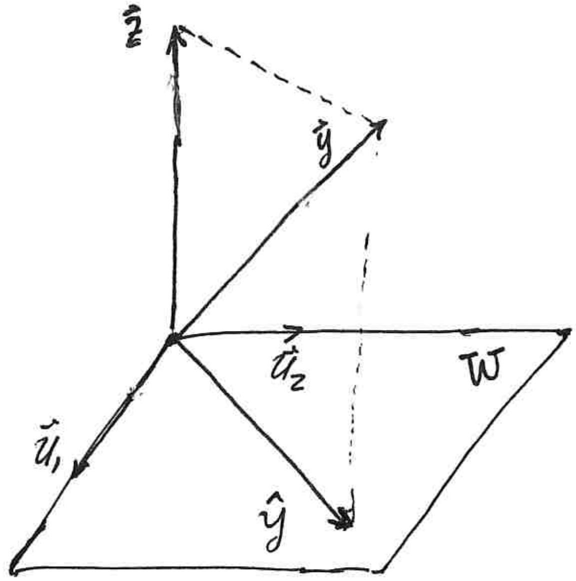
this is equivalent to ask for

$\vec{z} = \vec{y} - \hat{y}$ to be orthogonal to

any $\vec{w} \in W$.

Now, from previous thm $\vec{z}$ is $\perp$ to any vector in W if and only if $\vec{z} \cdot \vec{u}_j = 0$, for all $\vec{u}_j$

From here follow the proof in 9.



$$\vec{y} = \hat{y} + \vec{z}$$

Proof.- Consider an arbitrary $\vec{y} \in \mathbb{R}^n$

and define $\hat{y} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_p \vec{u}_p \in W$ (9.1)

and $\vec{z} = \vec{y} - \hat{y}$. (9.2)

Since we want $\vec{z} \in W^\perp$, we require $\vec{z}$ to be orthogonal to all $\vec{u}_j$ $(j = 1, .., p)$ (This is enough according to previous thm)

Thus,

$$(\vec{z} \cdot \vec{u}_j) = ((\vec{y} - \hat{y}) \cdot \vec{u}_j) = 0, \quad j = 1, .., p.$$

or equivalently

$$\vec{y} \cdot \vec{u}_j - ((\alpha_1 \vec{u}_1 + \cdots + \alpha_j \vec{u}_j + \cdots + \alpha_p \vec{u}_p) \cdot \vec{u}_j) = 0$$

using orthogonality

$$\alpha_j (\vec{u}_j \cdot \vec{u}_j) = \vec{y} \cdot \vec{u}_j$$

$$\Rightarrow \boxed{\alpha_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j},} \quad j = 1, 2, .., p. \qquad (9.3)$$

Therefore,

$\vec{z} = \vec{y} - \hat{y}$ is orthogonal to W

if $$\boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p = \mathrm{proj}_W \vec{y}} \quad (10.1)$$

As a Consequence,

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W$ is given by (10.1) and $\vec{z} = \vec{y} - \hat{y} \in W^\perp$.

<u>Uniqueness</u> If there exists $\hat{y}_1 \in W$ and $\vec{z}_1 \in W^\perp$ Such that $\vec{y} = \hat{y}_1 + \vec{z}_1$

then $\hat{y}_1 + \vec{z}_1 = \vec{y} = \hat{y} + \vec{z}$

$$\Rightarrow \hat{y}_1 - \hat{y} = \vec{z} - \vec{z}_1 \Rightarrow \hat{y}_1 - \hat{y} \in W \cap W^\perp$$

$$\Rightarrow \hat{y}_1 - \hat{y} = \vec{0} \Rightarrow \underline{\hat{y}_1 = \hat{y}}$$

$$\Rightarrow \vec{z} - \vec{z}_1 = 0 \Rightarrow \underline{\vec{z}_1 = \vec{z}}$$

<u>Def.</u> The vector $\hat{y}$ given by (10.1) is Called "the orthogonal projection of $\vec{y}$ onto W."

The notation for this is

$$\boxed{\hat{y} = \mathrm{proj}_W \vec{y}}$$

6.3 Exerc. #3

$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Verify $\{\vec{u}_1, \vec{u}_2\} \subset \mathbb{R}^3$ is an orthogonal set and find the orthogonal projection of $\vec{y}$ onto $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

Answ: $\vec{u}_1 \cdot \vec{u}_2 = [1\ 1\ 0] \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -1+1 = 0.$ ✓

According to thm 8

$$\text{proj}_W \vec{y} = \hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 =$$

$$= \frac{[-1\ 4\ 3] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1\ 1\ 0] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{[-1\ 4\ 3] \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{[-1\ 1\ 0] \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\Longrightarrow$

$$\text{proj}_W \vec{y} = \hat{y} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

and $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

and $\vec{y} = \overset{\in W}{\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}} + \overset{\in W^\perp}{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}} = \hat{y} + \vec{z}$ decomposition

<u>Theorem 10</u> $B = \{\vec{u}_1, .., \vec{u}_p\}$ is <u>an orthonormal basis</u> for $W \subset \mathbb{R}^n$.

Then

for $\vec{y} \in \mathbb{R}^n$,

a) $\boxed{proj_W \vec{y} = (\vec{y}.\vec{u}_1)\vec{u}_1 + (\vec{y}.\vec{u}_2)\vec{u}_2 + \cdot\cdot + (\vec{y}.\vec{u}_p)\vec{u}_p}$ (3.1)

b) If $U = [\vec{u}_1\ \hat{u}_2 \ldots \vec{u}_p]$, then

$$proj_W \vec{y} = U U^T \vec{y}.$$

<u>Proof:-</u>

a) By definition

$$proj_W \vec{y} = \frac{\vec{y}.\vec{u}_1}{\vec{u}_1.\vec{u}_1}\vec{u}_1 + \cdot\cdot + \frac{\vec{y}.\vec{u}_p}{\vec{u}_p.\vec{u}_p}\hat{u}_p$$

If B is an orthonormal basis $\Rightarrow$ $\vec{u}_i . u_i = 1$ and (3.1) follows.

b)
$$U^T\vec{y} = \begin{bmatrix} \vec{u}_1^T \rightarrow \\ \vdots \\ \vec{u}_p^T \rightarrow \end{bmatrix} \begin{bmatrix} \vec{y} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T \vec{y} \\ \vdots \\ \vec{u}_p^T \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{y} \\ \vdots \\ \vec{u}_p \cdot \vec{y} \end{bmatrix}$$

$$\Rightarrow U(U^T\vec{y}) = \begin{bmatrix} \vec{u}_1 & \cdot\cdot & \vec{u}_p \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \vec{u}_1 \cdot \vec{y} \\ \vdots \\ \vec{u}_p \cdot \vec{y} \end{bmatrix} =$$

$$= (\vec{u}_1 \cdot \vec{y})\vec{u}_1 + \cdot\cdot (\vec{u}_p \cdot \vec{y})\vec{u}_p \overset{\text{Same as (3.1)}}{=\!=} proj_W \vec{y}.$$

Exercises:

Orthog. projections

Consider the plane

$W: x - y + 2z = 0$ and $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$

Find $\text{proj}_W \vec{v}$.

$x = t$
$y = s$
$z = \frac{s}{2} - \frac{t}{2}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ s \\ s/2 - t/2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

Answ: The vectors

$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $B^{\perp} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

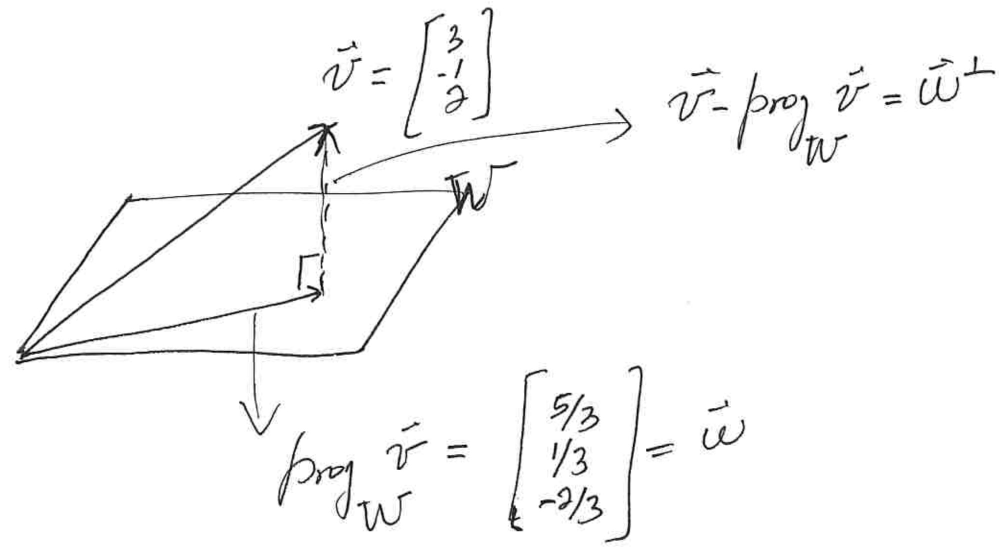
form an orthog. basis for W. (Verify it!).

Then,

$$\text{proj}_W \vec{v} \overset{\text{def}}{=} \left(\frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \left(\frac{\vec{v} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \right) \vec{u}_2$$

$$= \frac{2}{2} \vec{u}_1 - \frac{2}{3} \vec{u}_2 = \vec{u}_1 - \frac{2}{3} \vec{u}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} \checkmark$$



Then

$$\vec{w}^\perp = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$$

$\vec{w}^\perp \circ \vec{w} \equiv 0$

Verification: $\begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix} \circ \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = 20/3 - 4/3 - 16/3 = 0.$

From $\quad \vec{v} - \text{proj}_W \vec{v} = \vec{w}^\perp$

we get $\quad \vec{v} = \text{proj}_W \vec{v} + \vec{w}^\perp = \vec{w} + \vec{w}^\perp.$

or

$$\vec{v} = \overset{\text{in } W}{\begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}} + \overset{\text{in } W^\perp}{\begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}}$$

$$\vec{v} = \vec{w} + \vec{w}^\perp$$

$\checkmark$

<u>thm 5.10</u> A $m \times n$, then

$$(\text{row}(A))^{\perp} = \text{null}(A)$$

$$(\text{Col}(A))^{\perp} = \text{null}(A^T).$$

<u>Proof:-</u>

First, work on example 5.9.

$$A = \begin{array}{c} \begin{array}{ccccc} \vec{V}_1 & \vec{V}_2 & \vec{V}_3 & \vec{V}_4 & \vec{V}_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \sim \begin{array}{c} \begin{array}{ccccc} \vec{V}_1' & \vec{V}_2' & \vec{V}_3' & \vec{V}_4' & \vec{V}_5' \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_1' \\ r_2' \\ r_3' \\ r_4' \end{array} = B$$

Basis for $\text{row}(A) = ?$

To obtain basis for $\text{null}(A)$.

$$B\vec{x} = \vec{0} \iff \begin{aligned} x_4 &= -4x_5 \\ x_2 &= -3x_5 - 2x_3 \quad , \quad x_5, x_3 \text{ Free Vabs.} \\ x_1 &= x_5 - x_3 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_5 - x_3 \\ -3x_5 - 2x_3 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \overset{\vec{n}_1}{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_5 \overset{\vec{n}_2}{\begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}}$$

Basis for row $(A) = \{\vec{r}_1', \vec{r}_2', \vec{r}_3'\}$

Basis for null $(A) = \{\vec{n}_1, \vec{n}_2\}$

Verify that $\boxed{\vec{r}_i \cdot n_j = 0}$ $\quad i = 1, 2, 3$
$j = 1, 2.$

Then, $\left(\text{row}(A)\right)^{\perp} = \text{null}(A)$. why?

5.2

#12 Given

$$\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix} \text{ and } \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

and $W = \text{Span}(\vec{w}_1, \vec{w}_2)$

Find a basis for $W^{\perp}$.

Answ:

$$A = \begin{array}{c} \begin{matrix} \vec{w}_1 & \vec{w}_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 3 & -2 \\ -2 & 1 \end{bmatrix} \end{array}$$

$W = \text{Col}(A)$

$\Rightarrow \left(\text{Col}(A)\right)^{\perp} = \text{null}(A^T)$

$A^T\vec{x}=\vec{0}$ is equiv. to

$$\begin{bmatrix}1 & -1 & 3 & -2\\ 0 & 1 & -2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\sim\left[\begin{array}{cccc|c}1 & 0 & 1 & -1 & 0\\ 0 & 1 & -2 & 1 & 0\end{array}\right]\Rightarrow\begin{aligned}x_1&=x_4-x_3\\ x_2&=2x_3-x_4\end{aligned}$$

thus,

$$\vec{x}=\begin{bmatrix}x_4-x_3\\ -x_4+2x_3\\ x_3\\ x_4\end{bmatrix}=x_3\overset{\vec{n}_1}{\begin{bmatrix}-1\\ 2\\ 1\\ 0\end{bmatrix}}+x_4\overset{\vec{n}_2}{\begin{bmatrix}1\\ -1\\ 0\\ 1\end{bmatrix}}$$

<u>Basis for $W^\perp$</u> = <u>Basis for $\text{null}(A^T)$</u> = <u>$\{\vec{n}_1,\vec{n}_2\}$</u>